Green function for diffusion equation

- Consider the diffusion equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$

- Let's solve using the inverse Fourier transform,

$$u(x,t) = \int_{-\infty}^{\infty} u(k,t)e^{ikx}dx$$

- Substitution in the differential equation allows us to easily integrate the time dependence...

$$-k^2 u(k,t) = \frac{1}{\alpha^2}\frac{\partial u(k,t)}{\partial t}$$

- We obtain by integration $u(k,t) = u(k,t=t')e^{-\alpha^2 k^2 (t-t')}$
- We could also do it by separation of variables, practice it that way too....

Green function for diffusion equation, continued

- Assume we have a point source at $t = t'$, so that $u(x, t = t') = \delta(x - x')$
- We can then find $u(k, t = t')$ for the Fourier transform of the point source

$$u(k, t = t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x - x') e^{-ikx} dx = \frac{e^{-ikx'}}{2\pi}$$

- Finally we find $u(x, t)$ for $t > t'$ from the inverse Fourier transform

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} e^{-k^2\alpha^2(t-t')} dk$$

- The integral can be done by "completing the squares"

Green function for diffusion equation, continued

- The result of the integral is actually the Green function $G(x, x'; t, t')$

$$G(x, x'; t, t') = \frac{1}{[4\pi\alpha^2(t-t')]^{1/2}} e^{-(x-x')^2/4\alpha^2(t-t')}$$

- Notice that the Green function only depends on $x - x'$ and $t - t'$
- We find that at all times, $\int_{-\infty}^{\infty} G(x, x'; t, t')dx = 1$
- Then if we have a $t = 0$ distribution $u(x, t = 0)$, we can find $u(x, t)$ just by doing the integral,

$$u(x, t) = \int_{-\infty}^{\infty} G(x, x'; t, 0)u(x', t = 0)dx'$$