Chapter 30 – Inductance

- Mutual Inductance

- Self-Inductance and Inductors

- Magnetic-Field Energy

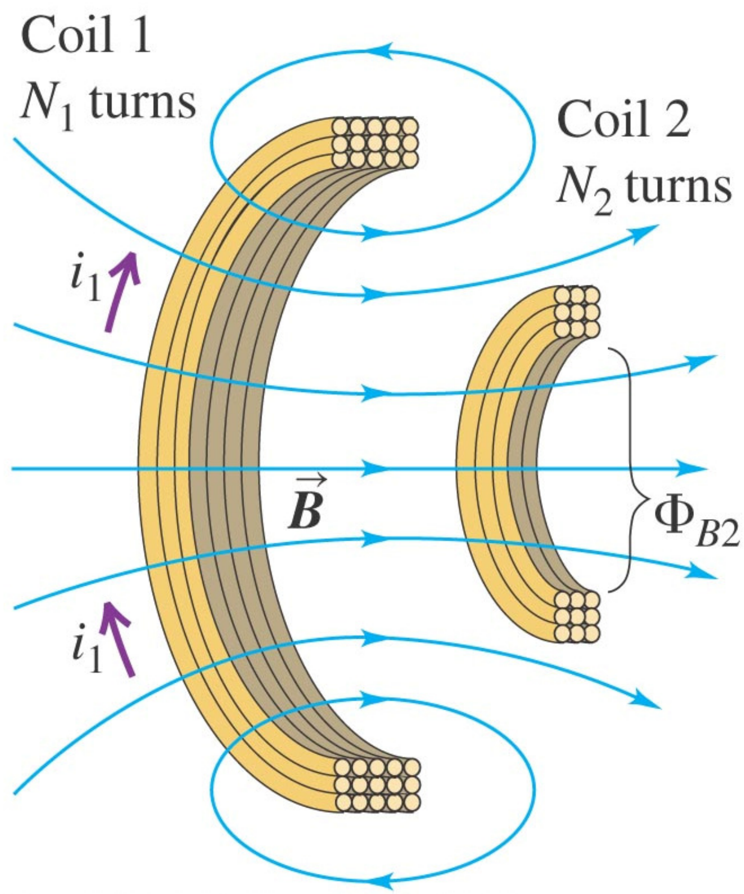
- The R-L Circuit

- The L-C Circuit

- The L-R-C Series Circuit

1. Mutual Inductance

- A changing current in coil 1 causes B and a changing magnetic flux through coil 2 that induces emf in coil 2.



$$\varepsilon_2 = -N_2 \frac{d\Phi_{B2}}{dt}$$

Magnetic flux through coil 2: $N_2\Phi_{B2} = M_{21} i_1$

Mutual inductance of two coils: M_{21}

$$N_2 \frac{d\Phi_{B2}}{dt} = M_{21} \frac{di_1}{dt}$$

$$\varepsilon_2 = -M_{21} \frac{di_1}{dt} \quad \rightarrow \quad M_{21} = \frac{N_2\Phi_{B2}}{i_1}$$

M_{21} is a constant that depends on geometry of the coils = M_{12}.

- If a magnetic material is present, M_{21} will depend on magnetic properties. If relative permeability (K_m) is not constant (M not proportional to B) → Φ_{B2} not proportional to i_1 (exception).

$$\varepsilon_2 = -M \frac{di_1}{dt}$$

$$\varepsilon_1 = -M \frac{di_2}{dt}$$

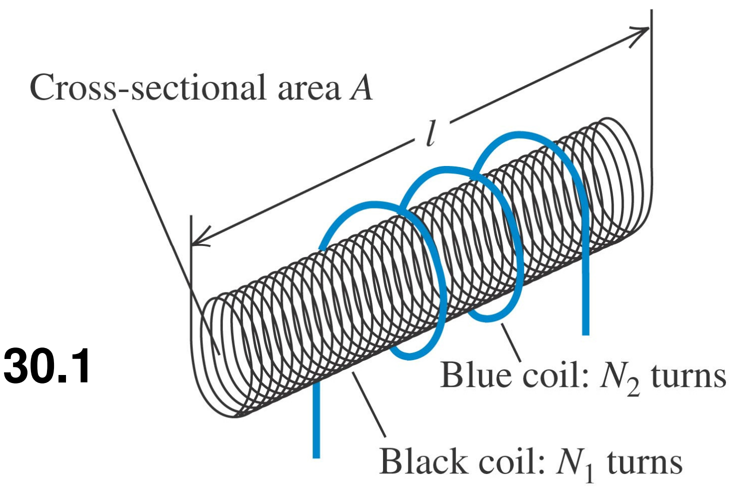
emf opposes the flux change

Mutual inductance: $$M = \frac{N_2 \Phi_{B2}}{i_1} = \frac{N_1 \Phi_{B1}}{i_2}$$

- Only a time-varying current induces an emf.

Units of inductance: 1 Henry = 1 Weber/A = 1 V s/A = 1 J/A^2

Ex. 30.1



2. Self Inductance and Inductors

- When a current is present in a circuit, it sets up B that causes a magnetic flux that changes when the current changes → emf is induced.

Lenz's law: a self-induced emf opposes the change in current that caused it → Induced emf makes difficult variations in current.

$$L = \frac{N\Phi_B}{i}$$

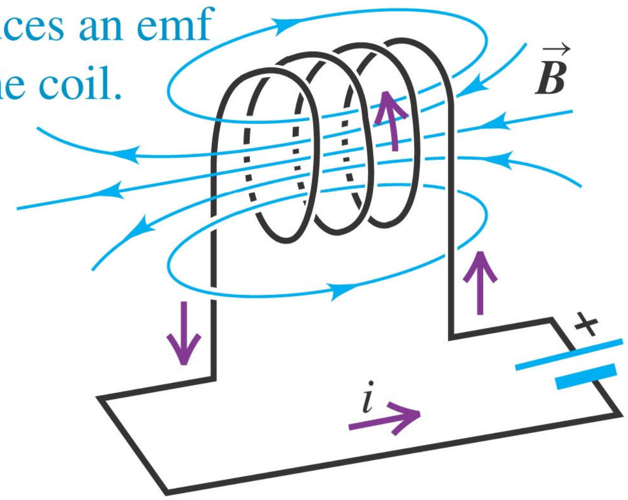
Self-inductance

$$N\frac{d\Phi_B}{dt} = L\frac{di}{dt}$$

$$\mathcal{E} = -L\frac{di}{dt}$$

Self-induced emf

Self-inductance: If the current i in the coil is changing, the changing flux through the coil induces an emf in the coil.



Inductors as Circuit Elements

Inductors oppose variations in the current through a circuit.

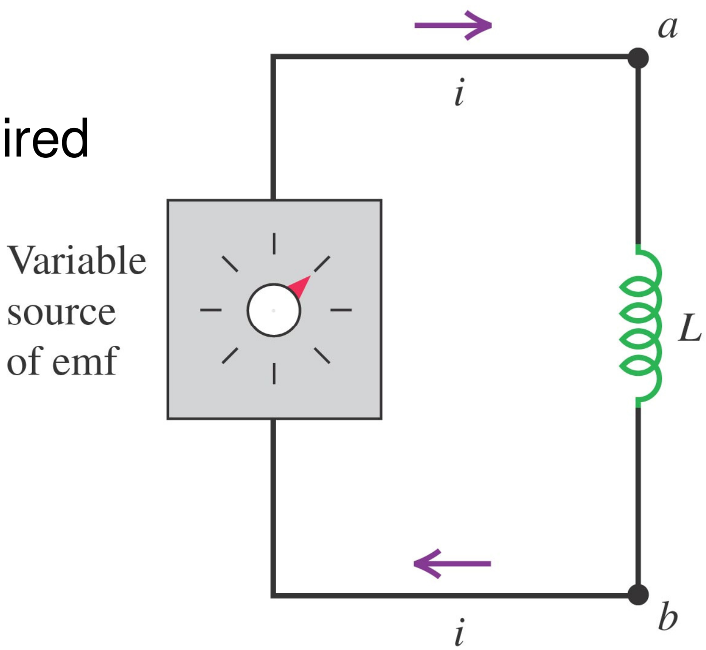
-In DC-circuit, L helps to maintain a steady current (despite fluctuations in applied emf). In AC circuit, L helps to suppress fast variations in current.

- Reminder of Kirchhoff's loop rule: the sum of potential differences around any closed loop is zero because E produced by charges distributed around circuit is conservative $\vec{E}_c$.

-The magnetically induced electric field within the coils of an inductor is non-conservative ($\vec{E}_n$).

- If R = 0 in inductor's coils → very small $\vec{E}$ required to move charge through coils → total $\vec{E}$ through coils $E_c + E_n = 0$. Since $E_c \neq 0$ → $E_c = -E_n$ accumulation of charge on inductor's terminals and surfaces of its conductors to produce that field.

$$\oint \vec{E}_n d\vec{l} = -L\frac{di}{dt}$$



- $E_n \neq 0$ only within the inductor.

$$\int_a^b \vec{E}_n \cdot d\vec{l} = -L\frac{di}{dt}$$

$$\vec{E}_c + \vec{E}_n = 0$$ (at each point within the inductor's coil)

$$\int_a^b \vec{E}_c \cdot d\vec{l} = L\frac{di}{dt}$$

- Self-induced emf opposes changes in current.

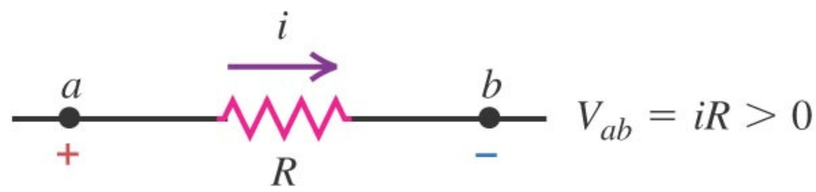
Potential difference between terminals of an inductor:

$$V_{ab} = V_a - V_b = L\frac{di}{dt}$$

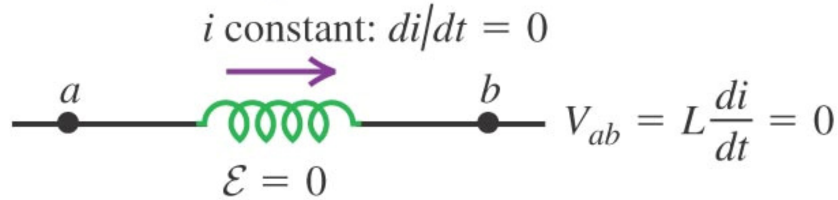
V_{ab} is associated with conservative, electrostatic forces, despite the fact that $\vec{E}$ associated with the magnetic induction is non-conservative → Kirchhoff's loop rule can be used.

- If magnetic flux is concentrated in region with a magnetic material → μ_0 in eqs. must be replaced by $\mu = K_m \mu_0$.

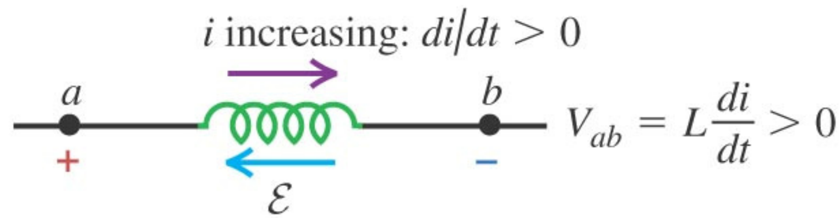
(a) Resisitor with current *i* flowing from *a* to *b*: potential drops from *a* to *b*.



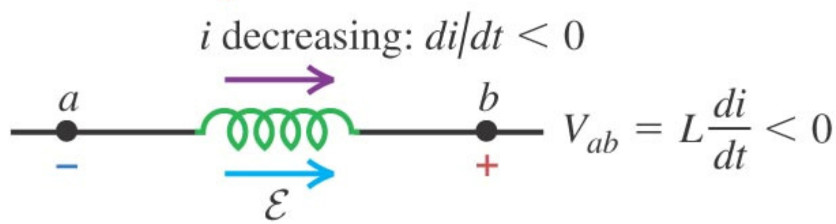
(b) Inductor with *constant* current *i* flowing from *a* to *b*: no potential difference.



(c) Inductor with *increasing* current *i* flowing from *a* to *b*: potential drops from *a* to *b*.



(d) Inductor with *decreasing* current *i* flowing from *a* to *b*: potential increases from *a* to *b*.



3. Magnetic-Field Energy

- Establishing a current in an inductor requires an input of energy. An inductor carrying a current has energy stored in it.

Energy Stored in an Inductor

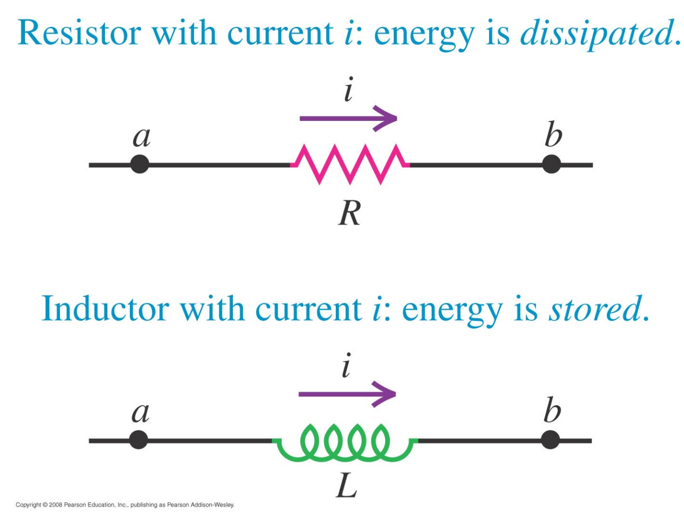
Rate of transfer of energy into L: $P = V_{ab} i = L \cdot i \cdot \dfrac{di}{dt}$

Energy supplied to inductor during dt: dU = P dt = L i di

Total energy U supplied while the current increases from zero to I:

$$U = L\int_0^I i \cdot di = \frac{1}{2} L I^2$$

Energy stored in an inductor



- Energy flows into an ideal (R = 0) inductor when current in inductor increases. The energy is not dissipated, but stored in L and released when current decreases.

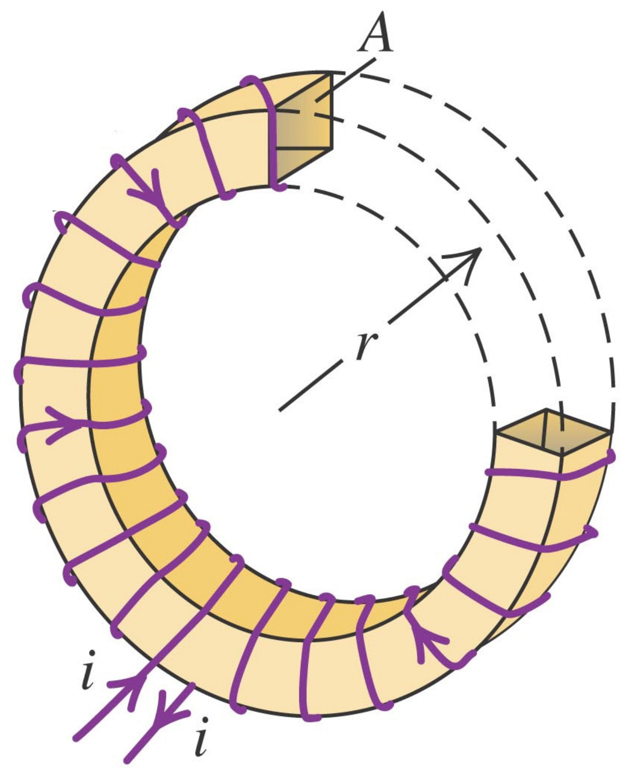
Magnetic Energy Density

-The energy in an inductor is stored in the magnetic field within the coil, just as the energy of a capacitor is stored in the electric field between its plates.

Ex: toroidal solenoid (B confined to a finite region of space within its core).

V = (2πr) A

$$L = \frac{\mu_0 N^2 A}{2\pi \cdot r}$$



$$U = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\mu_0 N^2 A}{2\pi \cdot r} I^2$$

Energy per unit volume: u = U/V

magnetic energy density

$$u = \frac{U}{V} = \frac{U}{2\pi \cdot r \cdot A} = \frac{1}{2} \mu_0 \frac{N^2 I^2}{(2\pi \cdot r)^2}$$

$$B = \frac{\mu_0 NI}{2\pi \cdot r}$$

$$\frac{N^2 I^2}{(2\pi \cdot r)^2} = \frac{B^2}{\mu_0^2}$$

$$u = \frac{B^2}{2\mu_0}$$

Magnetic energy density in vacuum

$$u = \frac{B^2}{2\mu}$$

Magnetic energy density in a material

4. The R-L Circuit

- An inductor in a circuit makes it difficult for rapid changes in current to occur due to induced emf.

Current-Growth in an R-L Circuit

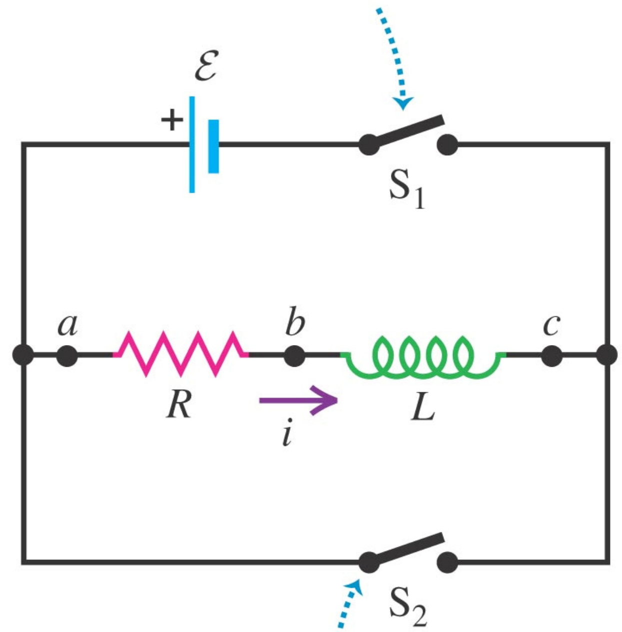
At t =0 → Switch 1 closed.

$$v_{ab} = I \cdot R \qquad v_{bc} = L\frac{di}{dt}$$

$$\varepsilon - i \cdot R - L\frac{di}{dt} = 0$$

$$\frac{di}{dt} = \frac{\varepsilon - iR}{L} = \frac{\varepsilon}{L} - \frac{R}{L}i$$

Closing switch S_1 connects the *R-L* combination in series with a source of emf $\mathcal{E}$.



Closing switch S_2 while opening switch S_1 disconnects the combination from the source.

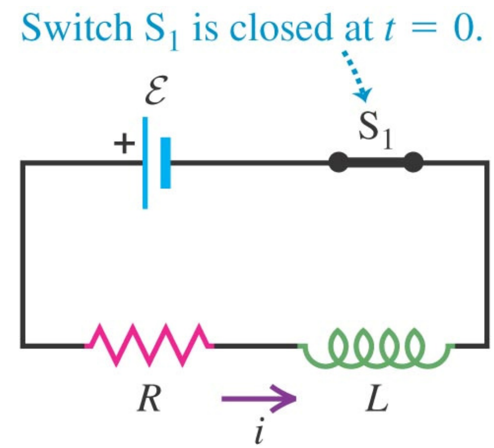
$$\left(\frac{di}{dt}\right)_{initial} = \frac{\varepsilon}{L}$$

(t = 0 → i = 0 → V_{ab} = 0)

$$\left(\frac{di}{dt}\right)_{final} = 0 = \frac{\varepsilon}{L} - \frac{R}{L} I$$

$$I = \frac{\varepsilon}{R}$$

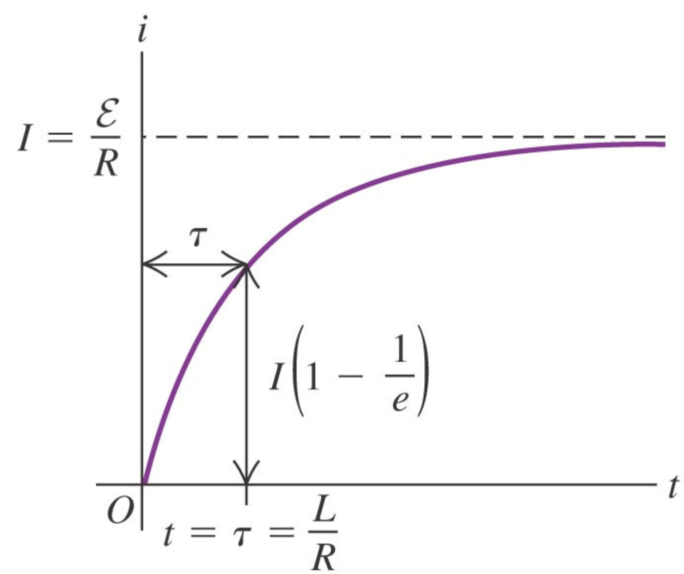
(t_f → di/dt = 0)



$$\frac{di}{i - \left(\varepsilon/R\right)} = -\frac{R}{L} dt$$

$$\int_0^i \frac{di'}{i' - (\varepsilon/R)} = -\int_0^t \frac{R}{L} dt'$$

$$\ln\left(\frac{i - \left(\varepsilon/R\right)}{-\varepsilon/R}\right) = -\frac{R}{L} t$$



Current in R-L circuit:

$$i = \frac{\varepsilon}{R}\left(1 - e^{-\left(R/L\right)t}\right)$$

$$\frac{di}{dt} = \frac{\varepsilon}{L} e^{-\left(R/L\right)t}$$

t = 0 → i = 0, di/dt = ε / L

t = ∞ → i → ε/R, di/dt → 0

Time constant for an R-L circuit:

$$\boxed{\tau = \frac{L}{R}}$$

At t = τ, the current has risen to (1-1/e) (63 %) of its final value.

Power supplied by the source:

$$\varepsilon \cdot i = \boxed{i^2 R} + \boxed{L \cdot i \frac{di}{dt}}$$

Power dissipated by R

Power stored in inductor

Current-Decay in an R-L Circuit

$$i = I_0 e^{-\left(R/L\right)t}$$

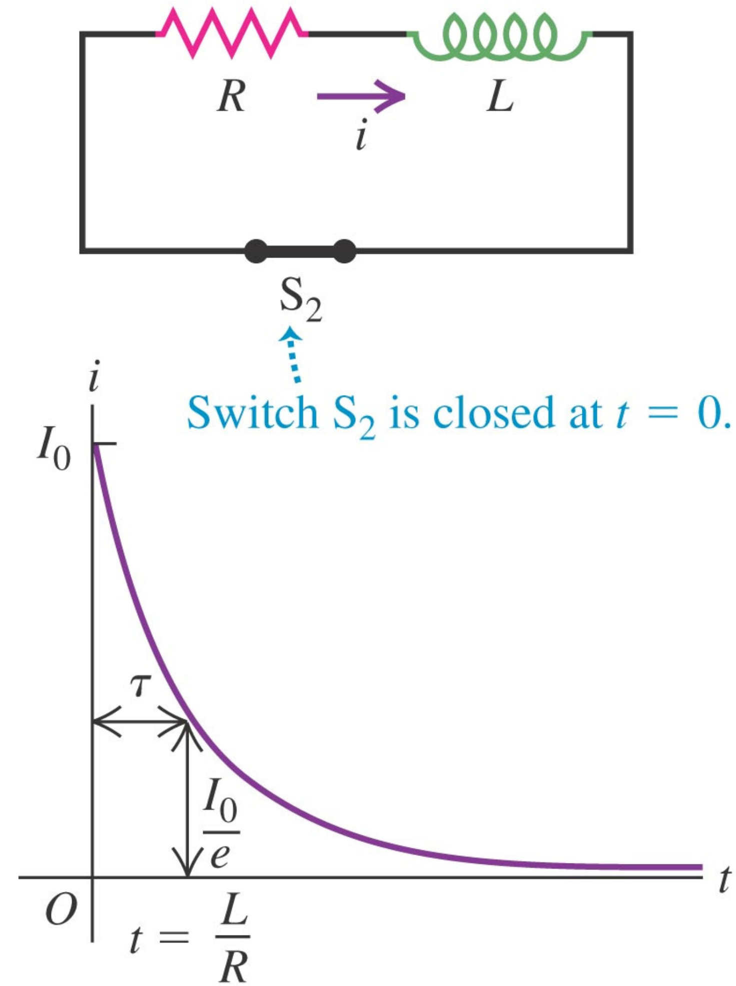
t = 0 → I_0

T = L/R → for current to decrease to 1/e (37 % of I_0).

Total energy in circuit:

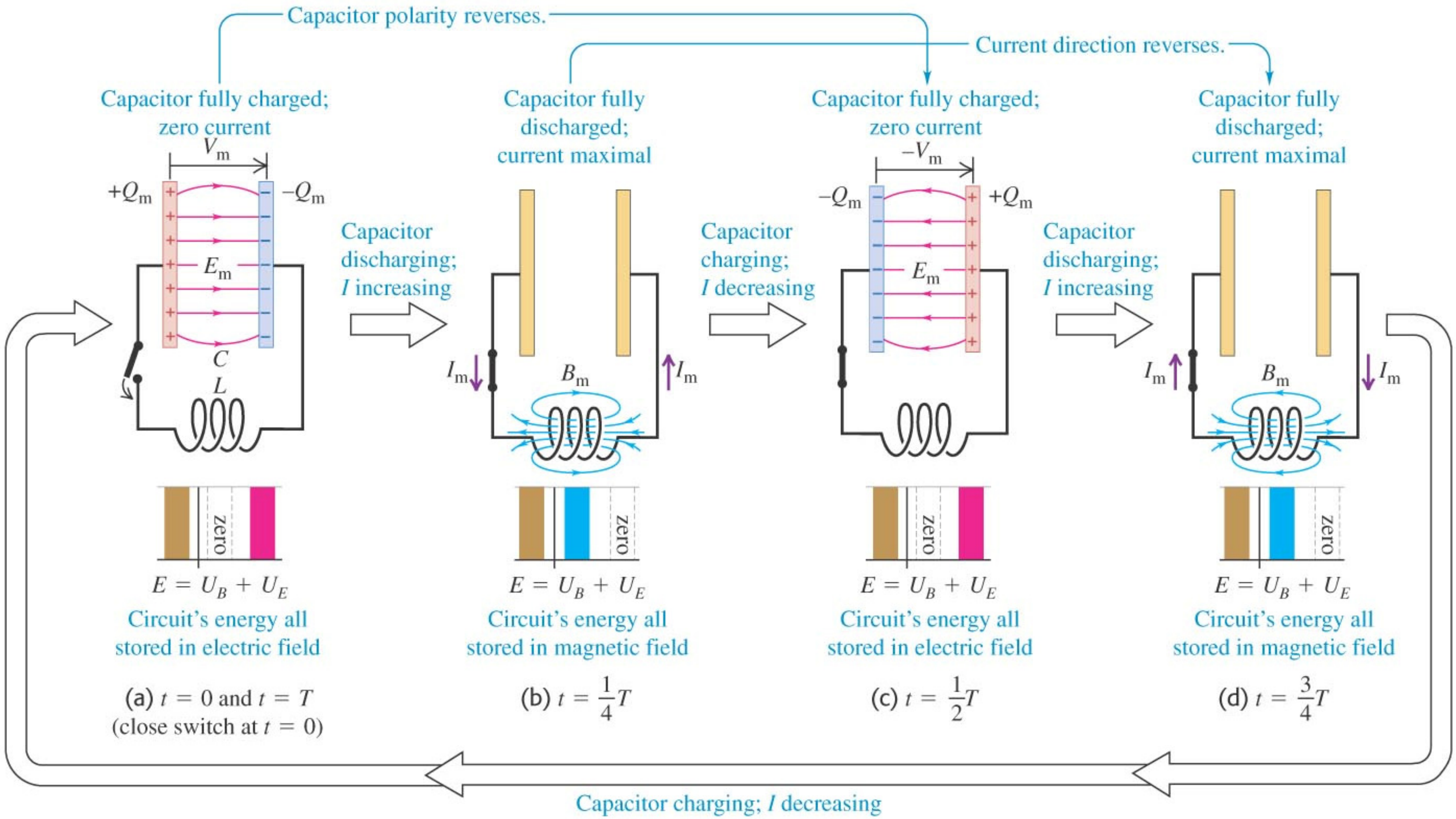
$$0 = i^2 R + Li\frac{di}{dt}$$

No energy supplied by a source (no battery present)



5. The L-C Circuit

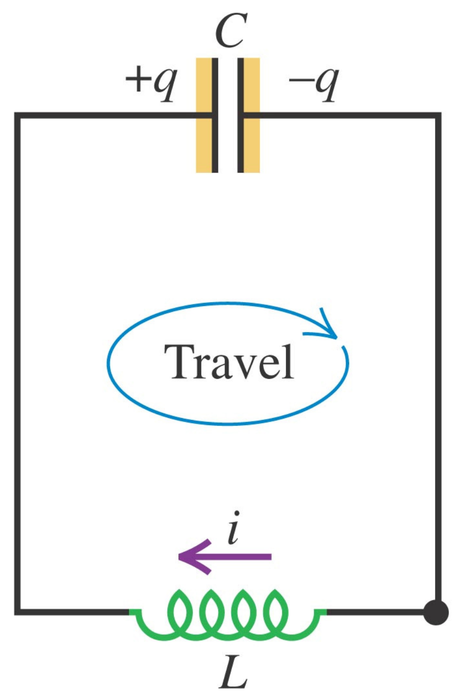
- In L-C circuit, the charge on the capacitor and current through inductor vary sinusoidally with time. Energy is transferred between magnetic energy in inductor (U_B) and electric energy in capacitor (U_E). As in simple harmonic motion, total energy remains constant.



L-C Circuit

- t = 0 → C charged → Q = C V_m
- C discharges through inductor. Because of induced emf in L, the current does not change instantaneously. I starts at 0 until it reaches I_m.
- During C discharge, the potential in C = induced emf in L. When potential in C = 0 → induced emf = 0 → maximum I_m.
- During the discharge of C, the growing current in L leads to magnetic field →energy stored in C (in its electric field) becomes stored in L (in magnetic field).
- After C fully discharged, some i persists (cannot change instantaneously), C charges with contrary polarity to initial state.
- As current decreases → B decreases → induced emf in same direction as current that slows decrease in current. At some point, B = 0, i =0 and C fully charged with $-V_m$ (-Q on left plate, contrary to initial state).
- If no energy loses, the charges in C oscillate back and forth infinitely → electrical oscillation. Energy is transferred from capacitor E to inductor B.

Electrical Oscillations in an L-C Circuit



Shown is + i = dq/dt (rate of change of q in left plate).
If C discharges → dq/dt<0 → counter clockwise "i" is negative.

Kirchhoff's loop rule: $$-L\frac{di}{dt}-\frac{q}{C}=0$$

$$\boxed{\frac{d^2q}{dt^2}+\frac{1}{LC}q=0}$$

L-C circuit

Analogy to eq. for harmonic oscillator: $$\frac{d^2x}{dt^2}+\frac{k}{m}x=0$$

$$x=A\cos(\omega t+\varphi)$$

$$\boxed{q=Q\cos(\omega t+\varphi)}$$

$$\boxed{i=\frac{dq}{dt}=-\omega\cdot Q\sin(\omega t+\varphi)}$$

$$\boxed{\omega=\sqrt{\frac{1}{LC}}}$$

Angular frequency of oscillation

ω = 2π f

If at t = 0 → Q_{max} in C, i = 0 → φ = 0

If at t = 0, q=0 → φ = ± **π/2 rad**

Energy in an L-C Circuit

Analogy with harmonic oscillator (mass attached to spring):

E_{total} = 0.5 k A^2 = KE + U_{elas} = 0.5 m v_x^2 + 0.5 k x^2 (A = oscillation amplitude)

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

Mechanical oscillations

- For L-C Circuit: $$\frac{Q^2}{2C} = \frac{1}{2}Li^2 + \frac{q^2}{2C}$$

Total energy initially stored in C = energy stored in L + energy stored in C (at given t).

$$i = \pm\sqrt{\frac{1}{LC}}\sqrt{Q^2 - q^2}$$

Electrical oscillations

Mass-Spring System

Kinetic energy $= \frac{1}{2}mv_x^2$

Potential energy $= \frac{1}{2}kx^2$

$\frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$

$v_x = \pm\sqrt{k/m}\sqrt{A^2 - x^2}$

$v_x = dx/dt$

$\omega = \sqrt{\dfrac{k}{m}}$

$x = A\cos(\omega t + \phi)$

Inductor-Capacitor Circuit

Magnetic energy $= \frac{1}{2}Li^2$

Electric energy $= q^2/2C$

$\frac{1}{2}Li^2 + q^2/2C = Q^2/2C$

$i = \pm\sqrt{1/LC}\sqrt{Q^2 - q^2}$

$i = dq/dt$

6. The L-R-C Series Circuit

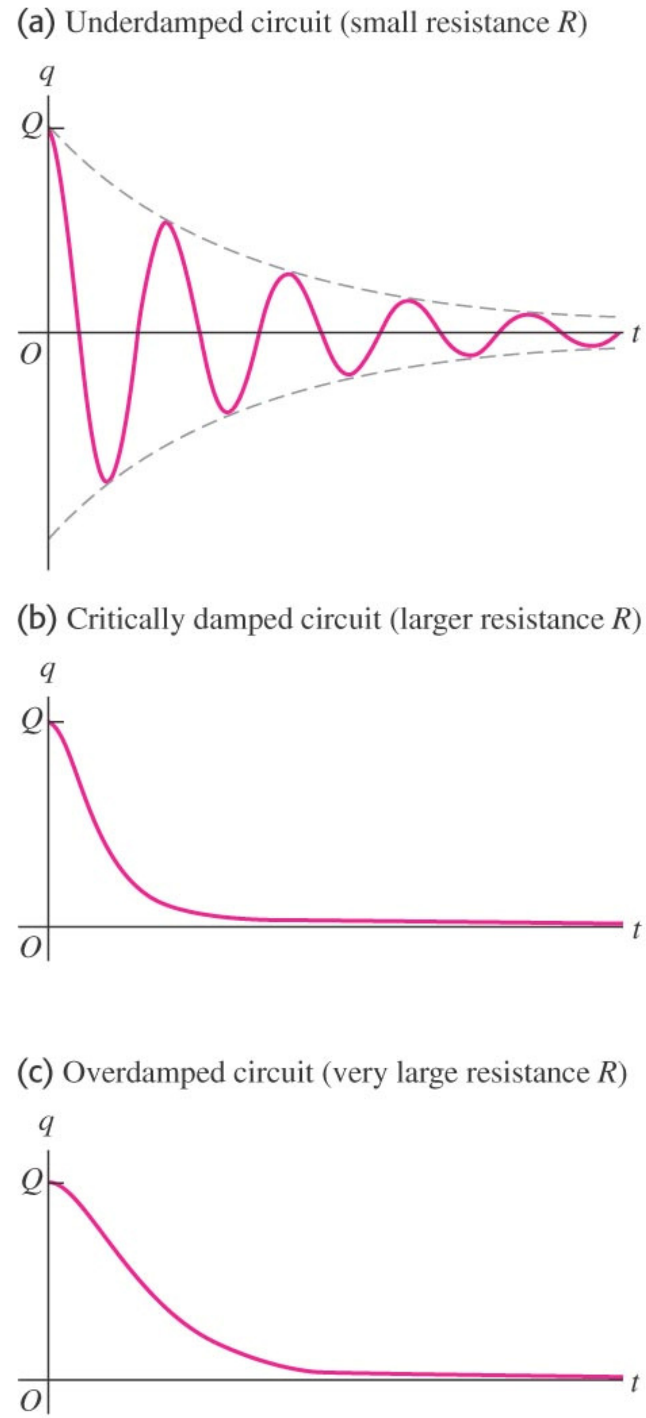
Because of R, the electromagnetic energy of system is dissipated and converted to other forms of energy (e.g. internal energy of circuit materials). [Analogous to friction in mechanical system].

- Energy loses in R → $i^2 R$ → U_B in L when C completely discharged < $U_E = Q^2/2C$

- Small R → circuit still oscillates but with "damped harmonic motion" → circuit underdamped.

- Large R → no oscillations (die out) → critically damped.

- Very large R → circuit overdamped → C charge approaches 0 slowly.



Analyzing an L-R-C Circuit

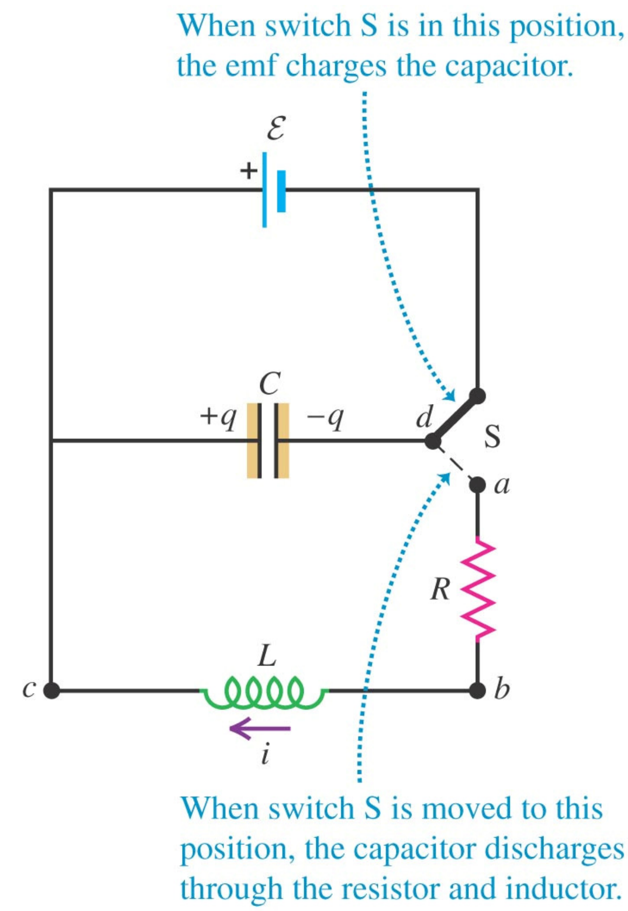
$$-iR - L\frac{di}{dt} - \frac{q}{C} = 0$$

(i = dq/dt)

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

For R < 4L/C: (underdamped)

$$q = Ae^{-(R/2L)t}\cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t + \varphi\right)$$



A, φ are constants

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

underdamped L-R-C series circuit

Inductors in series: $L_{eq} = L_1 + L_2$

Inductors in parallel: $1/L_{eq} = 1/L_1 + 1/L_2$